On the boundary of a union of rays

Panagiotis Alevizos, Jean-Daniel Boissonnat, Franco P. Preparata

INRIA

UNITÉ DE RECHERCHE
INRIA-SOPHIA ANTIPOLIS

Rapports de Recherche

N° 896

ON THE BOUNDARY OF A UNION OF RAYS

Programme 6

Panagiotis ALEVIZOS
Jean-Daniel BOISSONNAT
Franco P. PREPARATA

Septembre 1988

Institut National
de Recherche
en Informatique
et en Automatique

Domaine de Voluceau
Rocquencourt
B.P.105
78153 Le Chesnay Cedex
France
Tél.(1) 39 63 55 11

SUR LE BORD D'UNE UNION DE RAYONS

Panagiotis Alevizos
Jean-Daniel Boissonnat
Franco P. Preparata

Résumé

On étudie dans cet article le bord d'une union F de n demi droites (ou rayons) dans le plan. On présente deux résultats. Le premier est une nouvelle preuve, d'intérêt surtout méthodologique, du fait que le nombre d'arêtes d'une quelconque des composantes connexes de F est $O(n)$. Le deuxième résultat est un algorithme qui construit une telle composante connexe en temps optimal $\Theta(n \log n)$. Ces résultats sont obtenus grâce à l'introduction de la notion de squelette et d'une relation d'ordre associée qui peuvent présenter un intérêt dans d'autres contextes.

ON THE BOUNDARY OF A UNION OF RAYS*

Panagiotis Alevizos [†] **Jean-Daniel Boissonnat** [‡]
and Franco P. Preparata[§]

Abstract

In this paper, we study the boundary of a union F of n half lines (or rays) in the plane. We present two results. The first one is a novel proof of the $O(n)$ bound on the number of edges of any connected component of the boundary of F, which is essentially of methodological interest. The second is an algorithm for constructing such a connected component, which runs in optimal $\Theta(n \log n)$ time. A byproduct of our results are the notions of skeleton and of skeletal order, which may be of interest in their own right.

*This work was partly supported by the CEE ESPRIT Project P-940, by the Ecole Normale Supérieure, Paris, France, and by the NSF grant ECS-84-10902.

[†] Department of Mathematics, University of Patras, Greece and INRIA Route des Lucioles, 06565 Valbonne, France.

[‡] INRIA Route des Lucioles, 06565 Valbonne, France.

[§] University of Illinois at Urbana-Champain, IL 61801, USA. This work was done in part while this author was visiting the Ecole Normale Supérieure, Paris, France.

1 Introduction

Given is a set $R = \{r_1 ..., r_n\}$ of n rays (i.e., half lines) in the plane. To avoid insignificant degeneracies, we assume that the rays are in general position, i.e., all termini are distinct and no three rays intersect in the same finite point. The union $F = \bigcup_{j=1}^{n} r_j$ partitions the plane into a collection of regions, some of them being unbounded. Let $D(R)$ be this subdivision induced by F.

In the special case where the termini of all rays lie on a single straight line, Chazelle, Guibas and Lee [4] have shown that the boundary E of the region of $D(R)$ containing all the ray termini consists of $O(n)$ edges (whereas the total boundary of F may have $O(n^2)$ edges) and have given an $O(n \log n)$-time algorithm to compute it.

In [4], the alignment of the ray termini serves the purpose of readily supplying the order of the termini along E. The result of [4] that E has $O(n)$ edges is of a topological nature, and thus it is easily extended to the more general case, provided that the order of the termini along E is known. A recent paper of Alevizos, Boissonnat and Yvinec [2] gives a method for computing this order (hereafter referred to as the ABY-order) when all ray termini are known to belong to a single region of $D(R)$. The ABY-order is based on the following binary relation $\prec$ on a set of rays, exhibiting the property that all intersect the line l of equation $y = 0$ and all ray termini have positive ordinates : given two rays r_i and r_j, let x_i and x_j be the abscissae of the intersections of r_i and r_j with l respectively. Ray r_i precedes r_j, denoted $r_i \prec r_j$, if either r_i and r_j do not intersect in the half-plane $y > 0$ and $x_i < x_j$ or r_i and r_j intersect in the half-plane $y > 0$ and $x_i > x_j$. It is shown in [2] that this relation is a total order if and only if all ray termini belong to a single region of $D(R)$ and contains cycles otherwise.

This problem of finding the boundary of any region of $D(R)$ is closely related to the problem of finding the upper envelope (i.e. pointwise maximum) of n functions. This problem was first studied by Atallah [1] and then by Sharir, Hart and Wiernick [5,7]. For the case of n line-segments in the plane, none of which is vertical, Hart and Sharir have shown that the upper envelope consists of, at most, $O(n\alpha(n))$ edges where $\alpha(n)$ is an extremely slowly growing function related to the inverse Ackermann function. Their result extends immediately to the case of rays. Very recently, Wiernick and Sharir [7] proved that this bound is tight for line-segments, but, of course, it is not so for sufficiently long line-segments and, in particular, for rays.

The paper is organized as follows. Section 2 contains structural preliminaries for important subsets of rays satisfying the so-called "half-plane constraint". Section 2 introduces the notion of skeletal order, central to our technique, and contains a novel proof of the $O(n)$ bound on the number of edges of the boundary of any region of $D(R)$. In Section 3, we present an optimal algorithm to compute any such portion of the boundary in $O(n \log n)$ time. The half-plane constraint is dropped in Section 4, where structural and algorithmic results are presented for general sets of rays. Final remarks are presented in Section 5.

2 Structural preliminaries

2.1 Bundles

Assuming without loss of generality, that R contains no horizontal ray, there is a real value M_0 such that for any $M > M_0$, R is partitioned into two subsets, one, R^+, consisting of all rays intersecting the line $y = -M$, the other, R^-, consisting of all rays intersecting the line $y = M$. Our approach consists in investigating in Sections 2 et 3 the problem for R^+ and R^- separately and in combining the results to solve the original problem in Section 4. Thus, with a minor abuse of notation, until further notice, we let R denote a set with the property of R^+ defined above (referred to concisely as the *half-plane constraint*) and we let l denote the line $y = -M$. Note that in this case there is a unique unbounded region lying above $y = -M$, called *external region*, whose boundary is referred to as *external* and denoted E. All other regions of $D(R)$ are called *internal.*

Set $R = \{r_1, ..., r_n\}$ is partitioned into classes $R_1, ...R_h$ such that r_i and r_j are in the same class if and only if their respective termini lie in the same region of $D(R)$. The relation $\prec$ on a given class $R_h \subseteq R$ is a total order, by the result of [2]. We let t_i denote the terminus of r_i.

We consider the rays of a given class, say R_h, with $|R_h| = s$, and we re-index them so that $(r_1, ..., r_s)$ is their ABY-order. For convenience, we view this order as a string $\rho = r_1 r_2 ... r_s$. Given a substring $\alpha = r_i ... r_j$ of ρ, the external boundary of $\bigcup_{k=i}^{j} r_k$, called a *bundle*, consists of three portions : a polygonal chain C_I, called the *intermediate chain*, between the termini t_i of r_i and t_j of r_j, a polygonal chain C_L, called the *left chain*, between infinity and t_i, and a polygonal chain C_R, called the *right chain*, between infinity and t_j. It is immediate to realize that C_L and C_R are convex, and oppose their convex profiles. The bundle pertaining to α, denoted $b(\alpha)$, consists of chains C_L, C_I and C_R (see Figure 1); chains C_L and C_R are called the *external chains* of $b(\alpha)$.

We note that a single ray r with terminus t is itself a bundle, pertaining to a singleton string, with C_I degenerating to t and C_L and C_R both coinciding with the ray itself. We also note the following useful property.

Lemma 1 *Chains C_L and C_R are monotone with respect to a line orthogonal to l.*

Proof : By contradiction. Let $\alpha = r_i ... r_j$. Suppose that one convex chain of $b(\alpha)$, say C_L, is not monotone. Then there is a vertex p of C_L which is closest to l, and let r be the first ray whose segment in C_L violates monotonicity (see Figure 2).

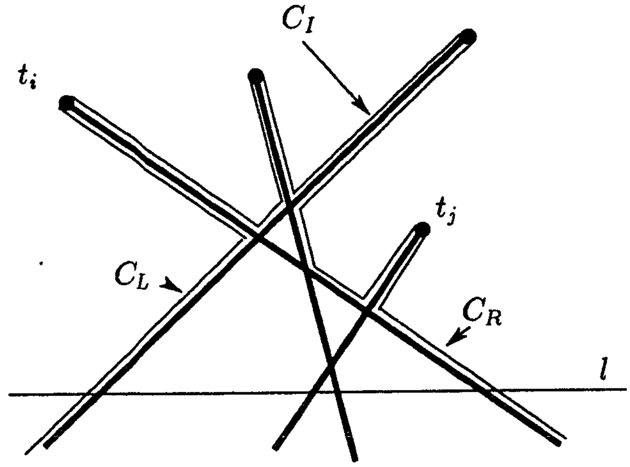


Figure 1: Illustration of the notion of $b(r_i, ..., r_j)$

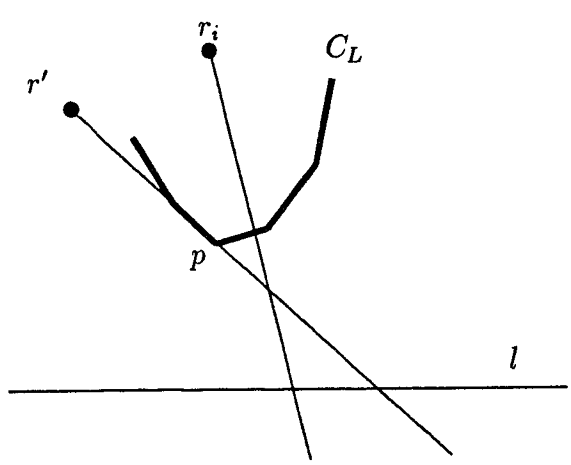


Figure 2: Chain C_L is monotone

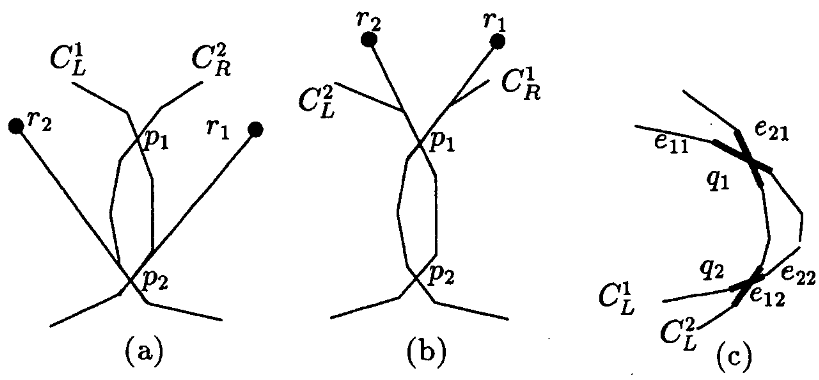


Figure 3: For the proof of Lemma 2 (Each pair of chains has at most one intersection point).

Due to the convexity of C_L, r precedes r_i in the ABY-order, contrary to the hypothesis that r_i is the leftmost term of string α. □

Two bundles are *disjoint* when they pertain to disjoint substrings. We now establish the following crucial property :

Lemma 2 *Any two external chains respectively belonging to two disjoint bundles intersect in at most one point.*

Proof: Let (C_L^i, C_I^i, C_R^i) be the chain of $b(\alpha_i)(i = 1,2)$, with $\alpha_1 \bigcap \alpha_2 = \emptyset$. Referring to their ABY-order, let α_1 be the left bundle.

First we observe that C_L^1 and C_R^2 intersect in at most one point. Indeed, suppose they intersect in more than one point. Then, due to their opposing convexities, they intersect in exactly two points, p_1 and p_2 with $y(p_1) > y(p_2)$ (see Figure 3a). Let r_1 and r_2 be the rays intersecting in p_2, with $r_1 \in \alpha_1$ and $r_2 \in \alpha_2$; according to the ABY-sorting rule given in the introduction, r_2 appears before r_1 in the ABY-order, contradicting the hypothesis.

Consider now the pair (C_R^1, C_L^2). They are both monotone with respect to the vertical (Lemma 1) and with opposing convexities; again they may intersect in at most two points p_1 and p_2 with $y(p_1) > y(p_2)$(see Figure 3b). Applying the same argument to the rays intersecting in p_1, we reach an analogous contradiction. Thus C_1^R and C_2^L intersect in at most one point (hereafter denoted with the letter "u").

Finally, consider the pair (C_L^1, C_L^2) (the pair (C_R^1, C_R^2) is treated analogously). Suppose that they intersect in more than one point. This means there are at least two intersections, and we consider two consecutive ones, q_1 and q_2. There are therefore four rays r_{11}, r_{12}, r_{21}, and r_{22}, such that r_{i1} and r_{i2} belong to $C_L^i(i = 1,2)$, and r_{1j} intersects $r_{2j}(j = 1,2)$(see Figure 3c).

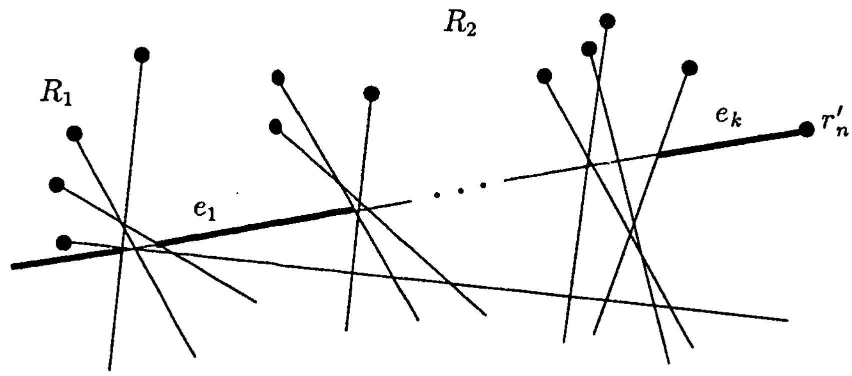


Figure 4: Subsets R_1 and R_2

It is immediate to realize that these rays are interlaced in the order (r_{11}, r_{21}, r_{12}, r_{22}), contrary to the disjointness hypothesis. Thus C_L^1 and C_L^2 intersect in at most one point (hereafter denoted with the letter "v_L"). □

2.2 Number of edges of the boundary of the regions of $D(R)$

2.2.1 Number of edges of the boundary of the external region

We make the conservative assumption that E passes through all ray termini. Indeed, let us suppose that a ray terminus is not on the external boundary; clearly, if we extend the ray so that it crosses the external boundary and its terminus appears on the external boundary, the number of edges of E increases. Thus the maximum number of edges of the external boundary is reached when all the ray termini are on the external boundary.

The rays form a string $\rho = r'_1...r'_n$ which represents their ABY-order. The ABY-order on the rays induces an order on the edges of the external boundary E. The attributes "before", "after" etc., as applied to edges of E, refer to this order.

Ray r'_n is referred to as the *last* ray of set R. Ray r'_n contributes $k \geq 1$ line segments e_1, ..., e_k of E (and possibly the initial half-line of E, in the ABY-order). We define R_1 as the subset of $R - \{r'_n\}$ consisting of all rays containing edges of E preceding e_1 (see Figure 4).

Clearly a substring $\alpha_1 = r'_1...r'_m$ of ρ is associated to R_1 and the complementary substring $\alpha_2 = r'_{m+1}...r'_n$ is associated to the subset $R_2 \triangleq R - R_1$. Thus the bundles $b(\alpha_1)$ and $b(\alpha_2)$ pertaining to α_1 and to α_2 are disjoint.

With this nomenclature, we are ready to prove the main theorem of this section :

Theorem 1 *The boundary of the external region of $D(R)$, for $R = \{r_1, ..., r_n\}$, has at most $4n - 2$ edges.*

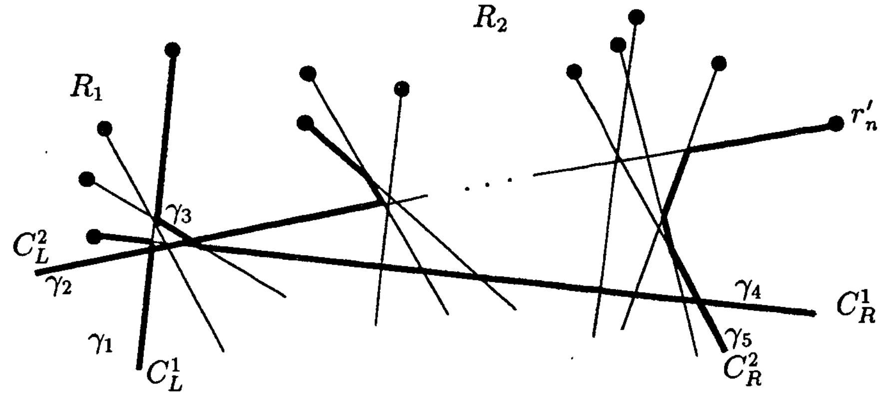


Figure 5: Intersections between E_1 and E_2

Proof : The proof is by induction. The theorem obviously holds for $n = 1$; we assume that it holds for any positive integer $t \leq n-1$.

Let us denote E_i the external boundary of the union of the rays of set R_i, and $|E_i|$ the number of edges of $E_i (i = 1,2)$. According to the inductive hypothesis, we have $|E_1| \leq 4m - 2$ and $|E_2| \leq 4(n-m) - 2$.

Let us denote (C_L^i, C_I^i, C_R^i) the chains of $b(\alpha_i)(i = 1,2)$. Because C_I^1 appears before e_1, it cannot intersect R_2 and similarly, because C_I^2 appears between e_1 and e_k, it cannot intersect R_1. Thus the only edges of E_1 and E_2 which intersect belong to right or left chains of the associated bundles.

Due to Lemma 2, these chains have at most three points of intersection. Specifically, we have at most (refer to Figure 5):

- an intersection between an edge of C_L^1, say γ_1, and the first edge of C_L^2, say γ_2, contained in r'_n.
- an intersection between an edge of C_R^1, say γ_3, and edge γ_2 of C_L^2.
- an intersection between an edge of C_R^1, say γ_4, and an edge of C_R^2, say γ_5.

The external boundary E of the union of all the rays $(R_1 \bigcup R_2)$ is the external boundary of the union of the external boundaries E_1 and E_2. It is clear, from the above discussion, that at most one edge of E_1 (γ_3) and at most one edge of E_2 (γ_2) may be cut into three pieces when merging the two bundles, two of them appearing as edges of E. Thus we have

$$|E| \leq |E_1| + |E_2| + 2 \leq 4n - 2$$

which completes the proof. □

2.2.2 Number of edges of an internal region of $D(R)$

Theorem 1 extends to the boundary of any internal region. This is obvious for those internal regions which do not contain a terminus. Let us consider an internal region whose boundary E_h contains the termini of $k \geq 1$ rays $r'_1, ..., r'_k$. We denote by C the smallest convex polygon bounded by rays (and, if applicable, by the line at infinity) which encloses the k ray termini. The boundary E_h consists of edges of C and of edges contained in rays $r'_1 ... r'_k$.

We partition the set of rays $R = \{r'_1, ..., r'_k\}$ into disjoint subsets consisting of all the rays intersecting C between two successive occurrences of edges of E_h contained in edges of C. Let m be the number of such subsets and n_i the number of rays pertaining to subset $i (i = 1, ..., m)$. Disjoint bundles are associated to these subsets and we have

$$|E_h| \leq \sum_{i=1}^{m} ((4n_i - 2) + |C| + m).$$

Indeed Theorem 1 holds for each bundle, and merging C and the bundles adds at most m edges to E_h. With $|C| < n$ and $\sum_{i=1}^{m} n_i < n$, we conclude $|E_h| < 5n - m$. Thus we have :

Theorem 2 *The boundary of any internal region of $D(R)$, for $R = \{r_1, ..., r_n\}$, has at most $O(n)$ edges.*

2.3 Legal descriptions

Let $R = \{r_1, \ldots, r_n\}$. In alternative cases, we shall augment R with an additional ray, either r_L – the line l with its terminus at $-\infty$ – or r_R – the same line l with its terminus at $+\infty$. Note that each ray r_i, $i = 1, ..., n$, intersects both r_L and r_R. We say that a string $\rho = r'_1 r'_2 \ldots r'_n$ (where $r'_1 r'_2 \ldots r'_n$ is a permutation of $r_1 \ldots r_n$) is a *legal description* of R or, briefly, *legal* if the subsequence of the rays belonging to a given class R_h of the partition of R yields their ABY-order. For example, the set R displayed in Figure 6 is partitioned into three classes: $\{r_6, r_8\}, \{r_5, r_4, r_{10}, r_9, r_1, r_2\}$ and $\{r_3, r_7\}$; $\rho = r_6 r_8 r_5 r_4 r_{10} r_9 r_7 r_3 r_1 r_2$ is a legal string for R.

We now introduce the notion of "skeleton" of a set R of rays. Let $r'_1 r'_2 \ldots r'_n$ be an arbitrary indexing of the members of R. We then have:

Definition 1 *The* skeleton *G of R pertaining to indexing $r'_1 r'_2 \ldots r'_n$ is constructively defined as follows: $G_0 = \{r_0\}$; G_i is obtained from G_{i-1} by adding to G_{i-1} the*

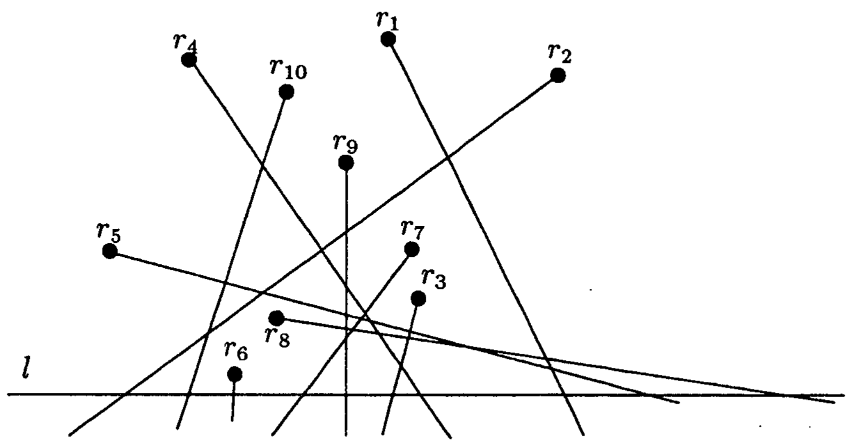


Figure 6: A set of rays

segment between the terminus of r'_i and its first intersection with G_{i-1}. Then, $G = G_n$.

Given an indexing $r'_1 r'_2 \dots r'_n$, Definition 1 constructively divides a ray r'_j into two portions, the *active* one (which is added to the skeleton), and its complement (the *inactive* portion, which may be empty). With this notion, we have the following alternative definition of skeleton:

Definition $1'$. *The* skeleton *G of R pertaining to indexing $r'_1 r'_2 \dots r'_n$ is constructively defined as follows: $G^*_0 = F$ (the union of the rays); G^*_i is obtained from G^*_{i-1} by removing from G^*_{i-1} the inactive portion of r'_i. Then, $G = G^*_n$.*

It is also convenient to view a ray r as an arbitrarily thin rectangle $\epsilon(r)$ (called the *ϵ-thickening* of r) obtained by an arbitrarily small translation of r orthogonal to r itself (see Figure 7). The boundary of $\epsilon(r)$ is conventionally directed as a *clockwise* circuit. This interpretation is naturally extended to the structures $G_0, G_1, \dots, G_n$ and $G^*_0, G^*_1, \dots, G^*_n$ introduced in Definitions 1 and $1'$; we let $\epsilon(G_i)$ denote the ϵ-thickening of G_i. (We recognize that the boundary of $\epsilon(G_n) = \epsilon(G^*_n) = \epsilon(G)$ is also the directed boundary of $\bar{G}$, the polygon obtained by removing from the plane the points of the skeleton G.) Examining in detail the transformations taking from G^*_{i-1} to G^*_i, we note that $\epsilon(G^*_i)$ is obtained by removing from $\epsilon(G^*_{i-1})$ a collection of trapezoids (see Figure 8), specifically, the points of $\epsilon(r'_i)$ corresponding to the inactive portion of r'_i and not shared by any other $\epsilon(r'_j)$, $j > i$. Finally, we note that the boundary of $\epsilon(G^*_0)$ is the collection of the directed boundaries of all the regions of $D(R)$ while $\epsilon(G)$ consists of a single circuit. It is a simple exercise to

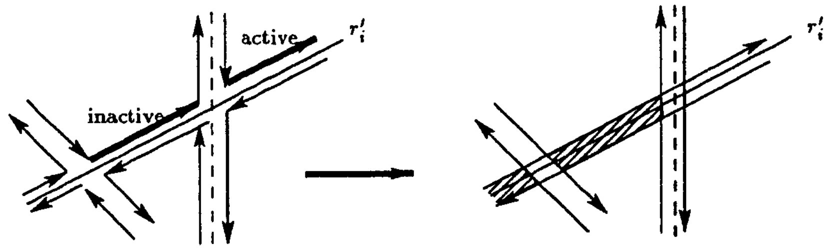


Figure 7: Illustration of the transformation from G_{i-1}^* to G_i^* in terms of their ϵ-thickenings.

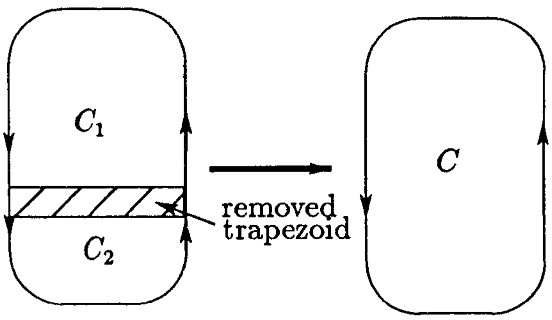


Figure 8: For the proof of Lemma 3

verify that G is a tree and that $\epsilon(G)$ has $4n + 3$ arcs.

Given a skeleton G, the *skeletal order* of G is the sequence of the ray termini in the left-to-right traversal of the boundary of $\bar{G}$. In the example of Figure 6, the skeletal order is $r_5r_4r_{10}r_9r_1r_2r_6r_8r_7r_3$.

We now link the notions of legal description and of skeletal order.

Lemma 3 *Let r' and r'' be two rays belonging the same class R_h. r' precedes r'' in the ABY-order if and only if r' precedes r'' in a skeletal order.*

Proof: Referring to the concepts introduced above, removal of an inactive portion trapezoid of $\epsilon(r_i')$ (while forming G_i^* from G_{i-1}^*) simply merges two boundary circuits C_1 and C_2 into a single circuit C_1 as illustrated in Figure 8.

Thus the removal of the trapezoid in question preserves the order of the termini, i.e., if r' preceded r'' in either C_1 or C_2, the same will hold in the resulting circuit C. Since the process of globally transforming G_0^* to G_n^* consists of the successive merging of pairs of boundary circuits, while preserving the order of the termini, the lemma follows. $\square$

We then have:

Theorem 3 *The skeletal order corresponding to an arbitray indexing of the rays of R is a legal description of R.*

We define the *leftist* skeleton G_L as the one corresponding to the order $(r_L, r'_1, ..., r'_n)$, and the *rightist* as the one corresponding to the order $(r_R, r'_n, ..., r'_1)$. The leftist (resp., rightist) skeleton has the property that the unique directed path from a ray terminus t to infinity is a polygonal line, monotone with respect to the vertical and containing only right (resp., left) turns.

3 Algorithmic issues

3.1 Lower bounds

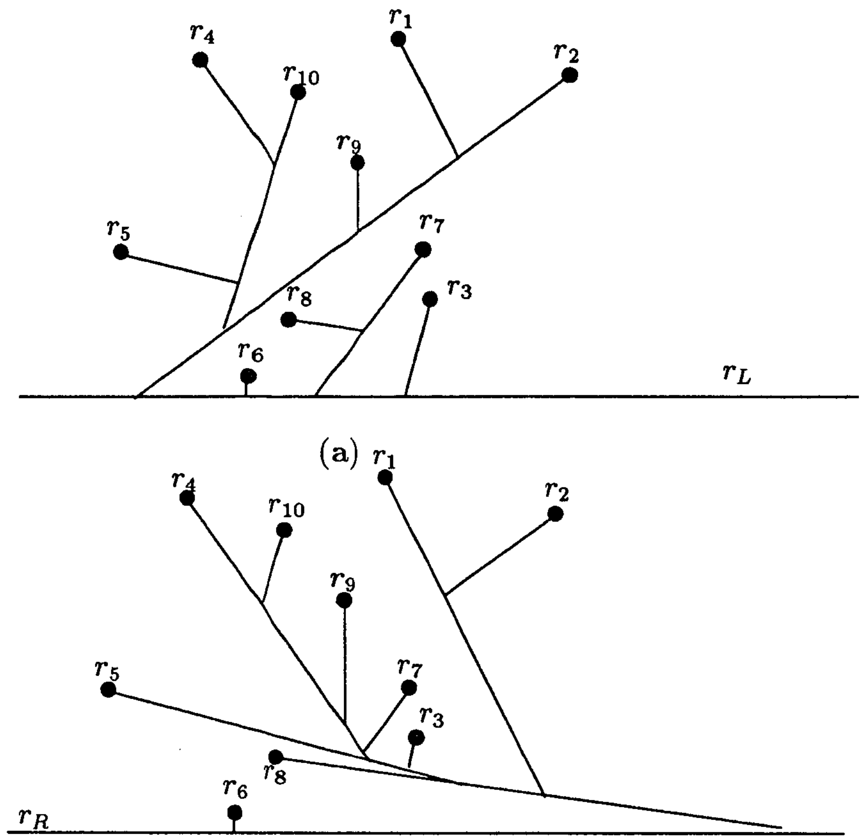
We observe that $\Omega(n \log n)$ is a lower bound for the time complexity of any algorithm for either the leftist or rightist order of a set $R = \{r_1, ..., r_n\}$. This result follows by transforming sorting to the problem in question. Indeed, let $x_1, ..., x_n$ be n real numbers. For each x_i $(i = 1, ..., n)$, we construct the vertical ray with terminus $(x_i, 1)$ and extending to $y = -\infty$. The resulting set of rays is of the same type as R^- (introduced in Section 2.1) and its leftist and rightist orders are identical and give the natural order of $x_1, ..., x_n$.

An analogous argument transforms sorting of n real numbers to the construction of the boundary of any region of $D(R)$, thereby establishing $\Omega(n \log n)$ as a lower bound also for the latter problem.

3.2 Computation of the leftist and rightist orders

We first note that the leftist (resp., rightist) order is readily obtainable from the corresponding skeleton. Indeed, we have shown in Section 2.3, that for any skeleton G, the corresponding skeletal order is obtained by a left-to-right traversal of the boundary of its ϵ-thickening $\epsilon(G)$. Since, for a set of n rays, the latter has $4n + 3$ arcs, the order is obtained in linear time from a skeleton.

The leftist and rightist skeletons are easily computable by a sweep algorithm by decreasing ordinate over the set of termini. The algorithm for constructing the leftist (resp., rightist) skeleton is a simple modification of the classical algorithm for intersecting segments. Indeed, the only difference with respect to the original algorithm is that any time the sweep-line reaches an intersection q of two segments r' and r'', rather than reversing the order of r' and r'' on the sweep-line, we just delete from the segment set the ray branch below q which lies to the right (resp., left). Clearly this task visits at most $(n-1)$ intersection points, and spends $O(\log n)$ work at each visited intersection. The time performance of the technique is therefore



(b)

Figure 9: Leftist (a) and rightist (b) skeletons.

$O(n \log n)$ and the correctness is trivially established. For the set of Figure 6, the leftist and rightist skeletons are shown in Figure 9.

Theorem 4 *The leftist and rightist skeletons of $R = \{r_1, ..., r_n\}$ can be constructed in time $O(n \log n)$, and this is optimal.*

3.3 Computation of the order of the ray termini on E

The leftist and rightist skeletons jointly enable us to uniquely detect the termini lying on the external boundary E. Let "$<_L$" and "$<_R$" respectively denote the ordering relation in the leftist and rightist skeletal orders. We then have :

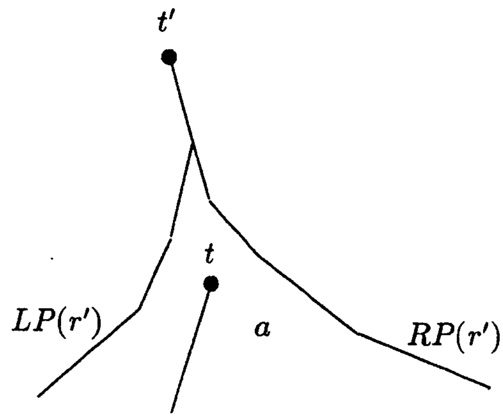


Figure 10: For the proof of Lemma 4

Lemma 4 *The terminus t of ray r does not lie on E if and only if there is at least one ray r', with terminus t' on E, such that $r' <_L r$ and $r <_R r'$.*

Proof : Let $LP(r)$ and $RP(r)$ respectively denote the paths from the terminus t of ray r to infinity in the leftist and rightist skeletons.

Sufficiency : There is a ray r' such that $r' <_L r$ and $r <_R r'$. Consider the polygonal lines $LP(r')$ and $RP(r')$. These two polygonal lines share a portion, a segment terminated by t'. They also delimit a portion a of the plane (see Figure 10), lying to the right of $LP(r')$ and to the left of $RP(r')$, to which t' does not belong. First, we notice that $y(t) < y(t')$. Indeed otherwise, $r' <_L r$ will imply that t' lies on the left of r while $r <_R r'$ will imply that t' lies on the right of r, a contradiction when the rays are in general position. When $y(t) < y(t')$, $r' <_L r$ means that t lies to the right of $LP(r')$, and $r <_R r'$ means that t lies to the left of $RP(r')$. We conclude that t belongs to region a. Since t is in general position, it does not lie on r', whence t and t' belong to distinct regions of the subdivision, i.e. t does not lie on E.

Necessity : Since t does not lie on E, then it belongs to an internal region of the subdivision. Let u be the maximum-ordinate vertex of this region and let r_1' and r_1'' be the two rays intersecting in u, with r_1' preceding r_1'' in the ABY-order. By the definition of the leftist and rightist skeletons, t lies to the left of $LP(r_1')$ and $LP(r_1'')$ and to the right of $RP(r_1')$ and $RP(r_1'')$: indeed, in the half-plane $y \leq y(u)$, both $LP(r_1')$ and $LP(r_1'')$ lie to the left of r_1'' and both $RP(r_1')$ and $RP(r_1'')$ lie to the right of r_1', and t lies between r_1'' and r_1'. It follows that $r_1' <_L r, r <_R r_1', r_1'' <_L r$ and $r <_R r_1''$.

Therefore if either t_1' or t_1'' lie on E, the lemma holds. Suppose that neither does. Then we apply the same argument to, say, t_1', and so on. By so doing, we construct a finite sequence $r'_1, r'_2, ..., r'_s$ such that (i) the terminal t'_s of r'_s lies on E; (ii) $r'_s <_L r'_{s-1} <_L ... <_L r'_1 <_L r$ and $r <_R r'_1 <_R r'_2 <_R ... <_R r'_s$. We conclude that $r'_s <_L r$ and $r <_R r'_s$. □

An immediate consequence is :

Corollary 1 *The leftmost term in $<_L$ and the rightmost term in $<_R$ lie on E.*

Let $L[1:n]$ and $R[1:n]$ denote the leftist and rightist orders, respectively. For an integer j, $L[j]$ and $R[j]$ are respectively the ray in the j-th position in the leftist and rightist orders; for a ray r, $L^{-1}[r]$ and $R^{-1}[r]$ are respectively r' s positions in the leftist and rightist orders. We claim that the following simple algorithm correctly identifies the subset $R_1 \subseteq R$ of the rays whose termini lie on E :

1. **begin** $p^* := 0$;
2.   **for** $i = 1$ **to** n **do**
3.   **if** $R^{-1}[L[i]] > p^*$ **then**
4.   **begin** mark ray $L[i]$;
5. $p^* = R^{-1}[L[i]]$;
6.   **end**
7. **end.**

The marked rays form the set R_1. To establish the correctness of this simple scan, we consider the following invariant for $L[1:n]$:

1. for each $L[j] \in R_1 \bigcap L[1:i-1]$ we have $R^{-1}[L[j]] \geq j$;
2. if $j^* = \max\{j : L[j] \in R_1 \bigcap L[1:i]\}$, then $R^{-1}[L[h]] < R^{-1}[L[j^*]]$ for $j^* < h \leq i$.

This invariant certainly holds for $i = 1$, since $L[1]$ is marked, $R^{-1}[L[1]] \geq 1$, and there is no value of h verifying $j^* < h \leq 1$. Assume the invariant holds for $i-1$. Consider $L[i]$; $L[i]$ is marked (line 4) if and only if $R^{-1}[L[i]] > p^* = R^{-1}[L[j^*]]$. Indeed the two conditions $R^{-1}[L[i]] < p^*$ and $j^* < i$ are equivalent to $L[j^*] <_L L[i]$ and $L[i] <_R L[j^*]$, whence, by Lemma 3, $L[i]$ does not belong to R^* (and, correctly, it is not marked). Conversely, if $R^{-1}[L[i]] > R^{-1}[L[j^*]]$, then there is no h in $[1, i-1]$ such that $R^{-1}[L[h]] > R^{-1}[L[j^*]] > R^{-1}[L[i]]$. This trivially holds for $j \leq j^*$; for $j^* < h \leq i-1$, Invariant (2) establishes $R^{-1}[L[h]] < R^{-1}[L[j^*]] < R^{-1}[L[i]]$. Thus $L[i]$ is correctly added to R^* and (1) and (2) are easily established. Notice that the algorithm to extract R_1 from R runs in time proportional to $|R|$. Thus we conclude :

Theorem 5 *The order of the rays of $R = \{r_1, ..., r_n\}$ whose termini lie on E can be computed in time $O(n \log n)$, and this is optimal.*

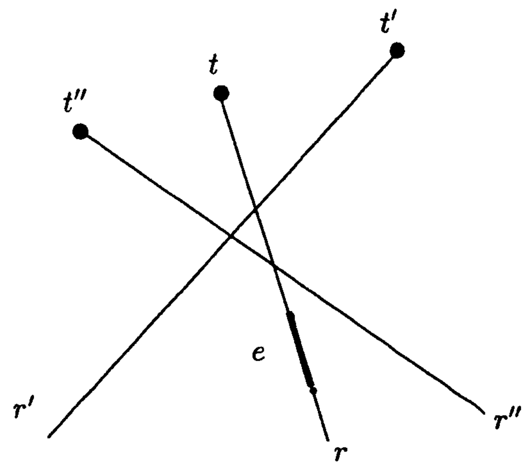


Figure 11: For the proof of Theorem 6

3.4 Construction of the external boundary

The construction of E is obtained from the leftist and rightist skeletons, G_L and G_R respectively. Let us consider the union $G_L \bigcup G_R$.

Theorem 6 *The external boundary of $G_L \bigcup G_R$ is identical to E.*

Proof : To prove the theorem, it is sufficient to prove that each edge of E belongs to an edge of G_L or to an edge of G_R (or both). Assume for a contradiction that e is an edge of E that does not belong to G_L nor to G_R. Let t be the terminus of the ray r containing e. If e does not belong to G_L, there necessarily exists a ray r' intersecting r between t and e whose terminus t' lies on the right side of the line containing r' (see Figure 11). Analogously, if e does not belong to G_R, there exists a ray r'' intersecting r between t and e whose terminus t'' lies on the left side of the line containing r'' . Thus e belongs to the wedge formed by r' and r'' and so cannot be an edge of the external boundary E, a contradiction. □

To construct the boundary of $G_L \bigcup G_R$ we use a plane sweep. This algorithm – closely related to the one used to compute either G_L or G_R – is again a minor modification of the classical algorithm for computing the intersections of segments. We sweep, by decreasing ordinates, the edges of $G_L \bigcup G_R$. The horizontal sweep-line status maintains a sequence of an odd number of intervals (possibly of width zero) which are alternatingly internal and external to the (unbounded) polygon whose boundary is E. Each event point in the sweep is either "internal" or "external", depending upon whether it is properly within an internal interval or not, respectively. At an internal point, no edge is inserted into the sweep-line status. An external

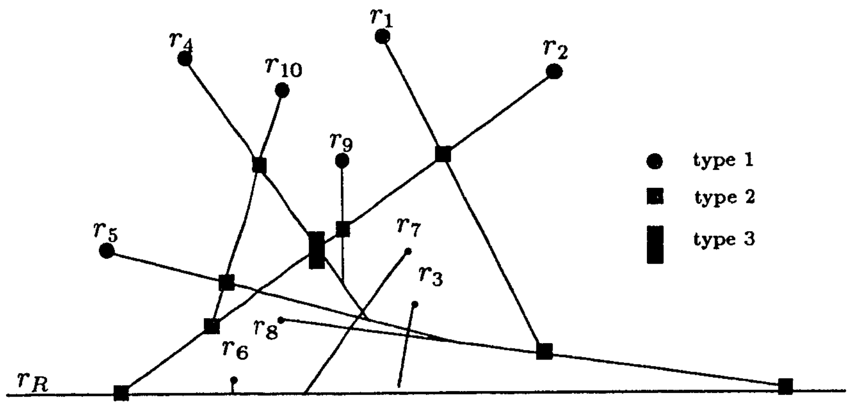


Figure 12: Illustration of the cases of the sweep algorithm for constructing E

point, however, requires an update of the sweep-line status, depending upon its classification as one of three types (each distinct case is illustrated in Figure 12) :

1. *A ray terminus.* Update : a 0-width interval is inserted into the sweep-line status (no edge of $G_L \bigcup G_R$ is eliminated)

2. *A vertex of* $G_L \bigcup G_R$ (but not a ray terminus). Calling the *indegree* of such vertex v the sum of the number of edges of G_L and of G_R incident on v from above, we distinguish the following subcases :

   (a) *The indegree of v is 3.* Update : a 0-width interval is deleted from the sweep-line status (i.e., it is merged with a nonzero-width interval) and one edge of $G_L \bigcup G_R$ (provably not on E) is eliminated.

   (b) *The indegree of v is 4.* Update : two 0-width intervals are merged into a single nonzero-width interval of the sweep-line status.

3. *The intersection q of an edge e_L of G_L and an edge e_R of G_R.* Update : nonzero-width intervals are merged into a single nonzero-width interval of sweep-line status. The portions of e_L and e_R below q (provably not on E) eliminated.

The correctness of the above technique is trivially established. As to its formance, since the number of edges of $G_L \bigcup G_R$ is at most $4n - 2$, we obtai $O(n \log n)$ time the sequence of the vertices of $G_L \bigcup G_R$, sorted by decreasing o nates. Since each edge of $G_L \bigcup G_R$ requires an $O(\log n)$ amount of processing. conclude :

Theorem 7 *The external boundary E can be computed in time $O(n \log n)$, and is optimal.*

3.5 Construction of the boundary of an internal region

Let A be an internal region, and let $R_1 \subseteq R$ be the set or rays with termini on the boundary of A. If R_1 is not empty, we denote by $D(R - R_1)$ the subdivision determined by $R - R_1$. Since the original subdivision is a refinement of this new subdivision, in the latter there is a unique region A_1, such that $A \subset A_1$. If in $D(R - R_1)$ region A_1 does not contain any ray terminus on its boundary, then it is a convex polygon and we are done. Otherwise let $R_2 \neq \emptyset$ be the set of rays whose termini are on the boundary of A_1. We remove them from $R - R_1$ to obtain a unique region A_2 of $D(R - R_1 - R_2)$ containing A_1 . This process is iterated, and we obtain a sequence of planar regions $A \subset A_1 \subset A_2 ... \subset A_s$ where A_s is a convex polygon (possibly unbounded). Notice that when $R_1 = \emptyset$, $A_s = A$. The boundary of A_s consists of two monotone chains C_L and C_R sharing a vertex v with largest ordinate in A_s. This boundary partitions the set R of rays into $\{R^{(1)}, R^{(2)}\}$, where $R^{(1)}$ consists of the rays whose terminus is internal to A_s (that is, $R^{(1)}$ is the union of the rays that have been successively removed in the constructions of the $A_1, A_2, ..., A_s$). This also shows that the ray termini on the boundary of A in $D(R)$ appear on boundary of the external region in $D(R^{(1)})$. Therefore, we can obtain the partition $\{R^{(1)}, R^{(2)}\}$ at the cost of $O(\log n)$ operations per ray if we know the boundary (C_L, C_R) of A_s. Its construction is our next objective.

Consider the vertex v, where C_L and C_R intersect. Vertex v is the intersection of two rays r' and r'', both in $R^{(2)}$; thus vertex v is common to the boundaries of all regions $A, A_1, ..., A_s$. In particular, v is the minimum ordinate intersection for all pairs of rays in $R^{(2)}$ whose lower wedge contains A, as is immediately shown by contradiction.

Let p be a point known to be in the interior of A (typically, p is a ray terminus). Given p and a ray $r \in R$, we say that r is a *left ray* (with respect to p) if p lies in the half-plane to the left of the line containing r, directed from the terminus t to infinity on r, and is a *right ray* otherwise.

Clearly, all edges of C_L belong to left rays, and all edges of C_R belong to right rays. Therefore, let $\{R_L, R_R\}$ be the partition of R into left and right rays (this partition is obtainable in $O(n)$ time), and let $r' \in R_L$ and $r'' \in R_R$ be the two rays intersecting at vertex v. It is clear that C_L belongs to the path originating at the terminus t of r' in $G_R(R_L)$ and C_R belongs to the path originating at the terminus of t'' of r'' in $G_L(R_R)$.

Therefore to construct the boundary of the region containing a given point p we proceed as follows:

1. Partition R into $\{R_L, R_R\}$ with respect to p; (* $T = O(n)$ *)

2. Construct $G_R(R_L)$ and $G_L(R_R)$. (* $T = O(n \log n)$ *)

3. Determine the lowest-ordinate intersection v with $y(v) > y(p)$ between $G_R(R_L)$

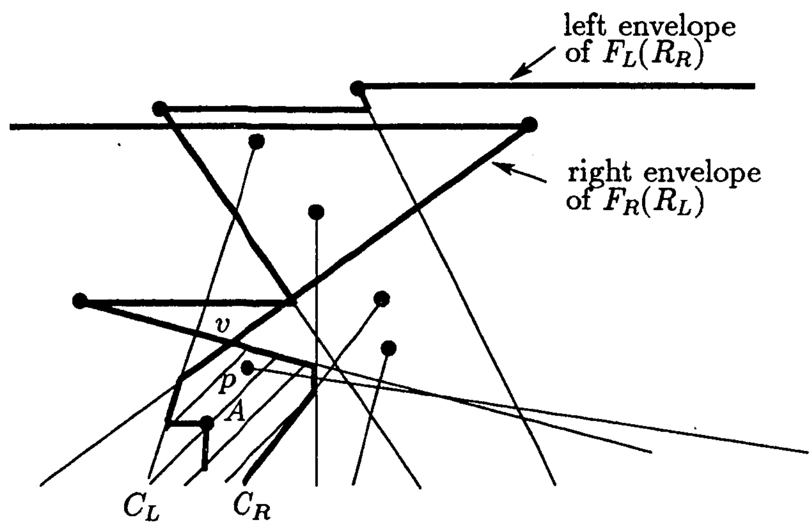


Figure 13: Left and right envelopes

and $G_L(R_R)$. From v, construct C_L and C_R; (* $T = O(n \log n)$, as shown below*)

4. Partition R into $\{R^{(1)}, R^{(2)}\}$; (* $T = O(n \log n)$ *)
If $R^{(1)} = \emptyset$, A is a trivial region, identical to the polygon (C_L, C_R). Else we perform two more steps :

5. Construct the boundary of the external region for set $R^{(1)}$, consisting of rays with terminus internal to the polygon (C_L, C_R). (* $T = O(n \log n)$ from Theorem 7 *)

6. Construct the intersection between the convex polygon (C_L, C_R) and the boundary of the external region of $D(R^{(1)})$. (* $T = O(n \log n)$ *)

We now show that Step 3 can be carried out in time $O(n \log n)$. We recall the definition of the envelope of a set of segments. The left shadow of a segment s is the portion of the horizontal slab determined by the extremes of s and lying to its left; the left shadow of a set of segments is the union of the left shadow of its members; the right envelope of a set of segments is the right boundary of its left shadow. The left envelope is analogously defined (refer to Figure 13).

Since the line $y = -M$ intersects all rays of R and leaves all termini on the same side, to determine v we just need to consider the right envelope of $G_R(R_L)$ and the left envelope of $G_L(R_R)$. A vertical plane-sweep by increasing ordinate constructs these envelopes in time $O(n \log n)$. Indeed, consider $G_R(R_L)$, and assume inductively that the sweep-line status contains the order of the intersections of $G_R(R_L)$ with the horizontal sweep-line. An event is either a terminus (when we suppress a term from the sweep-line status) or a branching (when we insert a term). The sequence of the rightmost terms appearing on the sweep-line status provides the right envelope. Clearly, this computation runs in time $O(n \log n)$.

Finally, the intersection of the two envelopes is analogous to merging two sorted sequences and is completed in additional $O(n)$ time (for $y \geq y(t)$). We conclude that the construction of the boundary of the region containing a given point p takes $O(n \log n)$ time. In particular, when p is a ray terminus, we have :

Theorem 8 *The boundary of any internal region of $D(R)$ can be computed in time $O(n \log n)$, and this is optimal.*

Note that the described method agrees with the original procedure for computing the external boundary. Indeed, in the latter case, v does not exist, and the role of the pair (C_L, C_R) is taken by the line at infinity.

4 The general case : structural and algorithmic results

In this section, we remove the half-plane constraint adopted at the beginning of Section 2 and held throughout Section 3, and consider an arbitrary set of rays.

Let us consider a region of the subdivision $D(R)$, identified by a point t in its interior, typically a ray terminus. Then, letting $n = |R|$, in time $O(n \log n)$ we can construct the boundaries E^- and E^+ of the regions respectively $D(R^-)$ and $D(R^+)$ which contain the chosen t. These are the boundaries of two simple polygons. The desired boundary E^* is the boundary of the connected component of the intersection of these two polygons which contains t.

In order to show that E^* has $O(|R|)$ edges, we prove the following lemma :

Lemma 5 *Let P and Q be two simple polygons with a total number of n edges and let $D(P \bigcup Q)$ be the subdivision induced by the union of the edges of P and Q. The boundary of any region of $D(P \bigcup Q)$ has at most $2n$ edges.*

Proof: : Let E^* be the boundary of a region of $D(P \bigcup Q)$. An edge of E^* is either contained in an edge of P or in an edge of Q. If an edge of E^* is contained in an edge e of P or Q, we label it with e. Thus we associate to E^* a circular sequence L of labels.

An edge of either P or Q will be called of type C_i if it contains i edges of E^*. Let γ_i be the number of such edges and m the maximum of i.

The following observation is due to Pollack, Sharir and Sifrony [6] : for any two edges e and f, L cannot contain a subsequence of the form $efefe$.

From the above observation, we deduce that among the edges of P (resp.Q) containing at least one edge of E^* and intersecting a given edge e of Q (resp. P), the only ones which may be of type $C_i, i > 1$, are the first one and the last one (when marching along E^*). Moreover these "extreme" edges cannot intersect e twice.

As a consequence, each edge if $e \in P$ (resp. Q) of type $C_j, j \geq 3$, Q (resp.P) contributes at most two edges of type $C_i, i > 1$, and at least $2j-4$ edges of type C_1. Thus

$$\gamma_1 \geq \sum_{j=3}^{m}(2j-4)\gamma_j.$$

Besides we have

$$\sum_{i=1}^{m}\gamma_i = n$$

and

$$\sum_{i=1}^{m} i\gamma_i = |E^*|.$$

Hence we conclude that

$$|E^*| \leq 2n - \frac{\gamma_1}{2} \leq 2n.$$

□

Boundary E^* can be computed in $O(n \log n)$. Indeed, Pollack, Sharir and Sifrony have shown that the contour of the union of two polygons can be obtained by applying the ray-shooting technique of Chazelle and Guibas [3] in time $O(\log n)$ per contour edge, i.e. in total time $O(n \log n)$ [6]. Their algorithm can be easily extended to compute a given connected component of the intersection of two simple polygons in the same time bound. Thus, we can state the main result of this paper :

Theorem 9 *The boundary of any region of the planar subdivision induced by n rays can be constructed in optimal time $O(n \log n)$.*

5 Concluding remarks

In conclusion, the main result of this paper is an optimal time algorithm for the construction of the boundary of any region of the subdivision determined by a set of n rays in the plane.

The fact – established in its full generality also in this paper – that the boundary of any such region has $O(n)$ edges draws a sharp conceptual distinction between sets of rays and sets of segments. Indeed, Wiernick and Sharir [7] have proved that the upper-envelope of the union of n line segments may have $O(n\alpha(n))$ edges where $\alpha(n)$ is the inverse Ackermann's function. Thus, Theorem 2 proves that this bound is not tight when the line segments are long enough.

The following proposition is a direct consequence of Theorems 2 and 10.

Proposition 1 *The boundary of any region of the planar subdivision induced by a set of n line segments, each intersecting a given line l, has $O(n)$ edges and can be constructed in $O(n \log n)$ time.*

Proof: Suppose without loss of generality that l is the x axis. Each line segment e_i has an end-point p_i^+ with positive ordinate and the other p_i^- with negative ordinate. Segment e_i can be considered as the intersection of two rays r_i^+ with terminus p_i^+ and r_i^- with terminus p_i^-. Let $R^+ = \{r_i^+ : i = 1, ..., n\}$ and $R^- = \{r_i^- : i = 1, ..., n\}$. Each region of the planar subdivision determined by the n segments is the disjoint union of two (not simultaneously empty) regions, respectively of $D(R^+) \bigcap (y \geq 0)$ and $D(R^-) \bigcap (y < 0)$. The boundary of any such region has $O(n)$ edges (by Theorem 2) and can be constructed in time $O(n \log n)$ by Theorem 10. This establishes the claim. □

Further research is needed to see if it is possible to extend our technique, most notably the notion of skeletal order, to the case of line segments in general position and to design an algorithm for computing the boundary of any region of the resulting arrangement.

Acknowledgments

Jean-Pierre Merlet is acknowledged for supplying to us his interactive drawing preparation system JPdraw .

References

[1] M. Atallah : Dynamic Computational Geometry, IEEE Symposium on Theory of Computing, 1983.

[2] P. Alevizos, J.D. Boissonnat, M. Yvinec : An optimal $O(n \log n)$ Algorithm for Contour Reconstruction from Rays, ACM Symposium on Computational Geometry, Waterloo, June 1987.

[3] B. Chazelle, L. Guibas : Visibility and Intersection Problems in Plane Geometry, ACM Symposium on Computational Geometry, Baltimore, June 1985.

[4] B. Chazelle, L. Guibas, D.T. Lee : The Power of Geometric Duality, *BIT* 25 (1985), 76-90.

[5] S. Hart, M. Sharir : Non Linearity of Davenport-Schinzel Sequences and of Generalized Path Compression Schemes, *Combinatorica* 6(2), 1986, pp 151-177.

[6] R. Pollack, M. Sharir, S. Sifrony : Separating Two Simple Polygons by a Sequence of Translations, Discrete Comp. Geom. 3:123-136 (1988).

[7] A. Wiernik, M. Sharir : Planar Realizations of Nonlinear Davenport - Schinzel Sequences by Segments, Discrete Comp. Geom. 3:15-47 (1988).

Imprimé en France
par
l'Institut National de Recherche en Informatique et en Automatique